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RESEARCH MEMORANDUM

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# PROJECT RAND

## RESEARCH MEMORANDUM

ON MULTI-STAGE GAMES WITH IMPRECISE PAYOFF

Richard Bellman

RM-1337

9 September 1954

Assigned to \_\_\_\_\_

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Summary: It is shown that under certain natural conditions the play of a multi-stage game is to a great extent independent of the payoff function.

## ON MULTI-STAGE GAMES WITH IMPRECISE PAYOFF

Richard Bellman

### §1. Introduction

There is a large class of situations of economic and military significance which can be considered to be multi-stage games. In many of these situations, the payoff function is easily determined; in others, it is a matter of difficulty to determine a suitable criterion.

The purpose of this note is to show, in a heuristic fashion, that in many cases the optimal play is independent of the precise form of the payoff, provided only that this payoff possesses certain intuitive properties.

### §2. Description of the Multi-stage Game

Let us consider a zero-sum multi-stage game where each play is determined by the game matrix  $A = (a_{ij})$ . Let initially the first player possess a quantity  $x$  of resources and the second player a quantity  $y$ . Since the game is zero sum, the state of the game at any time is described by  $x$ .

Defining a suitable criterion function, let  $f(x)$  represent the value of the game to the first player.

Then,  $f(x)$  satisfies the functional equation

$$\begin{aligned} f(x) &= \min_q \max_p \sum_{i,j} p_i q_j f(x + a_{ij}) \\ &= \max_p \min_q \sum_{i,j} p_i q_j f(x + a_{ij}) \end{aligned} \quad (2.1)$$

(see [1], [3]). The quantities  $p_i$  and  $q_j$  will, in general, depend upon  $x$ .

### §3. Assumptions Concerning $f(x)$

Let us now assume that  $x$  and  $y$  are large compared to  $a_{ij}$ . In other words, the state of the system is only slightly disturbed by any one play of the game. Furthermore, let us assume that the value of the matrix  $A$  is zero, which is to say, it is on the average a fair game. Otherwise, the play is relatively trivial.

Finally, we assume that it pays to start with a larger initial resource. Then

$$f'(x) > 0 \quad (3.1)$$

### §4. Heuristic Conclusion

Under these assumptions, we wish to show plausibly but not rigorously that the  $p_i$  and  $q_j$  are approximately independent of  $x$  and  $f(x)$ . This means that, under these assumptions, on each

play the players attempt to maximize the single-stage return, the ordinary expected value.

Let us write

$$f(x + a_{1j}) \approx f(x) + a_{1j}f'(x) \quad (4.1)$$

Then, from (2.1),

$$f(x) \approx \min_q \max_p \sum_{i,j} p_i q_j [f(x) + a_{1j}f'(x)] \quad (4.2)$$

or

$$f(x) \approx f(x) \sum_{i,j} p_i q_j + \min_q \max_p f'(x) \sum_{i,j} a_{1j} p_i q_j \quad (4.3)$$

whence, since  $f'(x) \neq 0$ ,

$$0 \approx \min_q \max_p \sum_{i,j} a_{1j} p_i q_j \quad (4.4)$$

$$\approx \max_p \min_q \sum_{i,j} a_{1j} p_i q_j$$

#### §5. Nonzero-sum Games

Let us consider a two-person, multi-stage, nonzero-sum game where the first and second players have, respectively, the game matrices

$$A = (a_{ij}), \quad B = (b_{ij}) \quad (5.1)$$

and initially the amounts  $x$  and  $y$ , respectively.

Let  $f(x,y)$  be some criterion function, such as probability of survival, assumed to satisfy the conditions

$$f_x > 0, \quad f_y < 0 \quad (5.2)$$

and assume that

$$\sum_{i,j} a_{ij} p_i q_j, \quad \sum_{i,j} b_{ij} p_i q_j < 0 \quad (5.3)$$

a game of attrition.

Then  $f(x,y)$  satisfies the functional equation

$$\begin{aligned} f(x,y) &= \max_p \min_q \left[ \sum_{i,j} p_i q_j f(x+a_{ij}, y+b_{ij}) \right], \quad x,y > 0 \\ &= \min_q \max_p \left[ \dots \right] \\ &= 1, \quad x > 0, \quad y < 0 \\ &= 0, \quad x < 0, \quad y > 0 \\ &= 1/2, \quad x = y = 0 \quad (\text{for the sake of completeness}) \end{aligned} \quad (5.4)$$

Assume as above that  $x$  and  $y$  are large compared to  $a_{ij}$  and  $b_{ij}$ . Then

$$f(x+a_{ij}, y+b_{ij}) \approx f(x,y) + a_{ij} f_x + b_{ij} f_y \quad (5.5)$$

Equation (5.4) then yields

$$\begin{aligned} 0 &\approx \max_p \min_q \left[ f_x \sum_{i,j} a_{ij} p_i q_j + f_y \sum_{i,j} b_{ij} p_i q_j \right] \\ &\approx \min_q \max_p \left[ \dots \right] \end{aligned} \quad (5.6)$$

or

$$\begin{aligned} \frac{f_x}{f_y} &\approx \min_p \max_q \left[ \frac{\sum_{i,j} a_{ij} p_i q_j}{\sum_{i,j} b_{ij} p_i q_j} \right] \\ &\approx \max_p \min_q \left[ \frac{\sum_{i,j} a_{ij} p_i q_j}{\sum_{i,j} b_{ij} p_i q_j} \right] \end{aligned} \quad (5.7)$$

This shows that the single-stage play is approximately governed by the criterion function

$$K(p,q) = \frac{\sum_{i,j} a_{ij} p_i q_j}{\sum_{i,j} b_{ij} p_i q_j} \quad (5.8)$$

That min-max = max-min in (5.7) is a result due to von Neumann. An elegant short proof based on the usual min-max theorem will be found in [4].

We thus have a rationale for the play of large classes of two-person nonzero-sum games.



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